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MODEL-COMPANIONS AND DEFINABILITY IN EXISTENTIALLY COMPLETE STRUCTURES[†]

BY

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ABSTRACT

The theory of model-companions and existentially complete structures is both reviewed and developed further. The review begins with A. Robinson's work in the fifties and continues through the definability of second order structures in existentially complete groups. New results include necessary and sufficient conditions for the existence of a model-companion in terms of the definability of general elementary properties. The main theorem of the paper gives necessary and sufficient conditions for the existence of a model-companion for universal theories with finite presentations and the amalgamation property. This result generalizes the result of P. Eklof and G. Sabbagh that the theory of R-modules has a model-completion if and only if R is coherent.

The metamathematics of algebra was one of Abraham Robinson's principal interests. This was the subject of his dissertation and the focus of his research in logic until the early sixties. During this period he published four books in this area: *On the Metamathematics of Algebra* (1951) [46], *Theorie Métamathématique des Îdéaux* (1955) [47], *Complete Theories* (1956) [48], and *Introduction to Model Theory and to the Metamathematics of Algebra* (1963) [49]. Two of his important contributions to this area were the concepts of model-completeness and model-completions. These ideas have led in time to the investigation of model-companions and the definability of general elementary properties and second order structures within existentially complete structures, the topic of this paper.

A structure $\mathfrak{M}$ is said to the *existentially complete for a theory* T if i) $\mathfrak{M}$ is a substructure of a model of T, and ii) each existential sentence which is defined in $\mathfrak{M}$, i.e., is in the language of $\mathfrak{M}$, and is true in a model of T which extends $\mathfrak{M}$ is

[†] The author's survey presentation at the Robinson Memorial Conference included the material in the first and fourth sections of this paper and Theorem 1 and its corollaries and examples in the second section. The other results in Section 2 and the results in Section 3 were obtained after the conference.

true in $\mathfrak{M}$ itself. The class of existentially complete structures for T will be denoted by $\mathscr{E}_T$. Some examples of existentially complete structures are listed in Table I.

TABLE I

Theory	Existentially Complete Structures for the Theory
Fields	Algebraically closed fields
Ordered fields	Real closed ordered fields
Discretely, nonarchimedean valued fields	Hensel fields
Differential fields	Differentially closed fields
Abelian groups	Divisible groups with infinitely many elements of each finite order
Groups	Algebraically closed groups
Commutative rings	Existentially complete commutative rings

An elementary property is a property which is defined by a single elementary formula. A *general elementary property* [5] is a property which is defined by a set of elementary formulas. Explicitly, a property P of n elements is a general elementary property (EP_Δ) if there is a set S of formulas $\varphi(v_1, \cdots, v_n)$ such that elements $a_1, \cdots, a_n$ of a structure $\mathfrak{M}$ have property P if and only if $\mathfrak{M}$ satisfies $\varphi(a_1, \cdots, a_n)$ for each formula φ in S.

A common feature of the first five examples above is that each general elementary property which is elementary for the existentially complete structures is elementarily determined for the models of the theory. The situation is quite different for the latter two examples. For the case of commutative rings, there is an elementary formula which is satisfied by an element of an existentially complete commutative ring if and only if that element is not nilpotent (a general elementary property). In the case of groups, there is an interpretation of second order number theory within the theory of algebraically closed groups.

The paper is divided into four sections. The first section, historical in nature, traces the development of the topic from A. Robinson's work in the early fifties. The second section discusses the relation between the definability of general elementary properties and the existence of a model-companion. These theorems are used in the third section to characterize a class of universal theories which have model-completions. The result of P. Eklof and G. Sabbagh on model-completions of R-modules is a consequence of this characterization. The fourth section describes, albeit briefly, the relevance of definable general elementary properties to theories without model-companions.

1. Historical survey

The origin of the recent work on definability in existentially complete structures can be identified as Abraham Robinson's investigations of model-completions in the fifties. A (first-order) theory T^* is said to be the *model-completion* of a (first-order) theory T if (0) T and T^* have the same language, (1) T^* includes T, (2) T and T^* are mutually model-consistent, and (3) $T^* \cup \mathrm{Diag}(\mathfrak{M})$ is a complete theory for each model $\mathfrak{M}$ of T ($\mathrm{Diag}(\mathfrak{M})$ is the set of all atomic sentences and negated atomic sentences which are defined and true in $\mathfrak{M}$). Equivalent formulations of (2) and (3) are (2′) each model of T is included in a model of T^* and vice-versa (of course, the vice-versa requirement is satisfied automatically since T^* includes T) and (3′) whenever extensions $\mathfrak{M}'$ and $\mathfrak{M}''$ of a model $\mathfrak{M}$ of T are models of T^*, then $\mathfrak{M}'$ and $\mathfrak{M}''$ are elementarily equivalent in the language of $\mathfrak{M}$. Some well-known examples of model-completions are listed in Table II.

TABLE II

Theory	Model-Completion
Fields	Algebraically closed fields
Ordered fields	Ordered, real closed fields
Discretely, nonarchimedian valued fields	Hensel fields
Differential fields	Differentially closed fields
Abelian groups	Divisible abelian groups with infinitely many elements of each finite order
Torsion-free abelian groups	Divisible, torsion-free abelian groups
Linear orderings	Dense linear orderings with neither first nor last element

Robinson demonstrated that model-completions lead to simple proofs of other important results. For example, one can deduce easily that the theory T_{ACF0} of algebraically closed fields of characteristic 0 is a complete, decidable theory admitting elimination of quantifiers from the facts that T_{ACF0} is the model-completion of the theory of fields of characteristic 0 and has a prime model, the algebraic closure of the rational numbers [49]. Model-completions lead also to easy proofs for Hilbert's Nullstellensatz [50], the existence of uniform bounds on the degree of the polynomials in Hilbert's Nullstellensatz [49], and Hilbert's seventeenth problem [49].

Nevertheless, the notion of a model-completion is not as inclusive as one would like, as Robinson noted. Although the theory of ordered fields has a model-completion, the related theory of formally real fields does not. The reason is the existence of real closed fields F_1 and F_2 such that $\sqrt{\sqrt{2}}$ is in F_1 and $\sqrt{-\sqrt{2}}$ is in F_2. Consequently, F_1 and F_2 are not elementarily equivalent in the language of their common subfield $Q(\sqrt{2})$ [49].

This example motivated the Eli Bers group to introduce model-companions. A theory T^* is the *model-companion* of a theory T if (0) T and T^* have the same language, (1) T and T^* are mutually model-consistent, and (2) T^* is model-complete. If a theory has a model-companion, then that model-companion is unique up to logical equivalence.

Clearly, a model-completion is also a model-companion, but the converse is false. The latter assertion is verified by the example of formally real fields. The theory of real closed fields is model-complete and is mutually model-consistent with the theory of formally real fields. Thus, the theory of real closed fields is a model-companion but not a model-completion for the theory of formally real fields.

In fact, the theory of formally real fields is the paradigm of a theory with a model-companion but not a model-completion. There is no model-completion because there is no formally real field F_3 such that

$$\begin{array}{ccccc} & & F_3 & & \\ & \subsetneq & & \supsetneq & \\ F_1 & & & & F_2 \\ & \supsetneq & & \subsetneq & \\ & & Q(\sqrt{2}) & & \end{array}$$

where F_1 and F_2 are the fields mentioned previously. In other words, F_1 and F_2 can not be amalgamated over $Q(\sqrt{2})$. A theory T is said to have the *amalgamation property* if whenever $\mathfrak{M}$, $\mathfrak{M}_1$, and $\mathfrak{M}_2$ are models of T and $\mathfrak{M}_1$ and $\mathfrak{M}_2$ extend $\mathfrak{M}$, then there is a model $\mathfrak{M}_3$ of T extending both $\mathfrak{M}_1$ and $\mathfrak{M}_2$. The following theorem of P. Eklof and G. Sabbagh [25] establishes the theory of formally real fields as the paradigm.

THEOREM A. *Assume that T^* is a model-companion for T and that T^* includes T. Then T^* is a model-completion for T if and only if T has the amalgamation property.*

This result brought two, related questions to the forefront: (1) What theories other than those in Table II, for example, groups or R-modules, have model-companions? (2) How can one prove that a theory does not have a model-companion? Both questions are ultimately questions about existentially complete structures, as the following theorem shows.

THEOREM B. *A theory T has a model-companion T^* if and only if the class $\mathscr{E}_T$ of existentially complete structures for T is a generalized elementary class* (EC_Δ), *in which case* $T^* = \mathrm{Th}(\mathscr{E}_T)$.

The first published proof that certain theories did not have model-companions appeared, to the best of my knowledge, in the paper "Model-completions and modules" by P. Eklof and G. Sabbagh [25]. They proved that the class $\mathscr{E}_T$ when T is either the theory of groups or the theory of modules over a noncoherent ring is not closed under ultrapowers and so cannot be a generalized elementary class. Specifically, they found general elementary properties which are elementary for the existentially complete structures but not for the theories. The existence of such general elementary properties entails the absence of a model-companion, as will be shown in the next section.

2. Definability and model-companions

The problem of existence of a model-companion and the problem of definability of general elementary properties for existentially complete structures are, for many theories, essentially two sides of the same coin. This coincidence will be spelled out in three steps. First, attention will be restricted to universal theories and general elementary properties corresponding to quantifier-free types. Then the coincidence will be discussed for arbitrary theories and general elementary properties. Finally, these results will be used in the next section to prove a generalization of the theorem of P. Eklof and G. Sabbagh on model-completions for modules.

Consider a language with variables $v_1, v_2, \cdots$. A (quantifier-free, universal, respectively) n-type Δ will be a set of (quantifier-free, universal, respectively) formulas in this language with free variables among $v_1, \cdots, v_n$. The conjunction, possibly infinitary, of the formulas in Δ will be denoted by $\wedge \Delta$.

DEFINITION 1. A formula $\varphi(v_1, \cdots, v_n)$ will be said to *generate* an n-type Δ for a class Σ of structures if each structure in Σ satisfies the sentence $\forall v_1 \cdots \forall v_n (\varphi \rightarrow \wedge \Delta)$. A formula will be said to *generate* an n-type for a theory T if it does so for the class $\mathrm{Mod}(T)$ of models of T.

DEFINITION 2. An n-type Δ will be said to be *self-generated* for a class Σ of structures if there is a finite conjunction of formulas from Δ which generates Δ

for Σ. An n-type will be said to be *self-generated* for a theory if it is self-generated for the class of models of the theory.

DEFINITION 3. An n-type Δ will be said to be *(existentially) principal* for a class Σ of structures if there is a (existential) formula $\varphi(v_1, \cdots, v_n)$ such that each structure in Σ satisfies the sentence $\forall v_1 \cdots \forall v_n\ (\varphi \leftrightarrow \wedge\Delta)$. The formula φ will be called a *principal generator* for Δ. An n-type Δ will be said to be *(existentially) principal* for a theory if it is (existentially) principal for the class of models of the theory.

A general elementary property corresponds to a type. The property is elementary if and only if that type is principal.

THEOREM 1. *If a theory has a model-companion, then each quantifier-free type which is existentially principal for the theory's existentially complete structures is self-generated for the theory itself.*

PROOF. Assume that T is a theory with a model-companion T^*. Suppose that Δ is a quantifier-free n-type which is existentially principal for the class $\mathscr{E}_T$ of existentially complete structures for T. Since $\mathscr{E}_T = \mathrm{Mod}(T^*)$, there is an existential formula φ such that $T^* \vdash \forall v_1 \cdots \forall v_n\ (\varphi \leftrightarrow \wedge\Delta)$. Consequently, there are formulas $\psi_1, \cdots, \psi_m$ in Δ such that $T^* \vdash \forall v_1 \cdots \forall v_n\ (\psi_1 \wedge \cdots \wedge \psi_m \rightarrow \varphi)$.

Let ψ_0 be the formula $\psi_1 \wedge \cdots \wedge \psi_m$. For each formula ψ in Δ, $T^* \vdash \forall v_1 \cdots \forall v_n$ $(\psi_0 \rightarrow \psi)$. Since $\forall v_1 \cdots \forall v_n\ (\psi_0 \rightarrow \psi)$ is a universal formula and T and T^* have precisely the same universal consequences, $T \vdash \forall v_1 \cdots \forall v_n\ (\psi_0 \rightarrow \psi)$ for each formula ψ in Δ. Hence, Δ is self-generated for T.

COROLLARY 1. *If a theory has a model-companion, then each quantifier-free type which is principal for the theory's class of existentially complete structures is self-generated for the theory itself.*

PROOF. Assume that T^* is the model-companion of a theory T. Since T^* is model-complete, each formula is equivalent with respect to T^* to an existential formula. Since $\mathscr{E}_T$ is just the class of models of T^*, each principal type for $\mathscr{E}_T$ is in fact existentially principal for $\mathscr{E}_T$.

COROLLARY 2. *If there is a quantifier-free type which is not self-generated for a theory but is existentially principal for the theory's class of existentially complete structures, then the theory does not have a model-companion.*

The usefulness of Corollary 2 is illustrated in the following examples.

EXAMPLE 1. Let T be the theory of groups. Let $\Delta = \{x^n \neq e : n > 0\}$, where e is the constant symbol in T for the identity element of a group (to simplify formulas, the symbols x, y, and z will be used in place of v_1, v_2, and v_3). This type is existentially principal for the class of existentially complete groups. A particular existential, principal generator for this type is the formula $\exists y\ \exists z \cdot (yx = x^2y \wedge zx^2 = x^2z \wedge zx \neq xz)$ [35, pp. 202–203, 222; or 72, pp. 20–21].

EXAMPLE 2. Let T be the theory of division rings of a specified characteristic p, a prime, or zero. Let $\Delta = \{q(x) \neq 0 : q(\xi)$ is a monic, irreducible polynomial over the prime field of the specified characteristic$\}$. An element realizes this type if and only if it is transcendental over the prime field. Clearly, this type is not self-generated for the theory T. However, it is existentially principal for the class of existentially complete division rings. A particular existential, principal generator for Δ is $\exists y\ \exists z\ (yx = x^2y \wedge zx^2 = x^2z \wedge zx \neq xz)$ [35, pp. 202–203; or 72].

EXAMPLE 3. Let T be the theory of commutative rings. Let $\Delta = \{x^n \neq 0 : n > 0\}$. An element realizes Δ if and only if it is not nilpotent. This type is not self-generated for the theory T. Nevertheless, it is existentially principal for the class of existentially complete commutative rings. A particular existential, principal generator for Δ is $\exists y\ \exists z\ (y^2 = y \wedge y \neq 0 \wedge xz = y)$ [17].

Theorem 1 has the following, partial converse.

THEOREM 2. *Assume that a universal theory T with at least one constant symbol has the amalgamation property. If each quantifier-free type which is existentially principal for the theory's class of existentially complete structures is self-generated for T, then T has a model-completion.*

PROOF. It suffices to show that the theory $\mathrm{Th}(\mathscr{E}_T)$ is model-complete, or, more specifically, that each existential formula is logically equivalent with respect to $\mathrm{Th}(\mathscr{E}_T)$ to a quantifier-free formula.

Let $\varphi(v_1, \cdots, v_n)$ be an existential formula. Let $\nabla(\varphi) = \{\psi(v_1, \cdots, v_n) : \psi$ is a universal formula and $T \vdash \varphi \to \psi\}$. It is well known that the existentially complete structures for T satisfy the formula $\forall v_1 \cdots \forall v_n\ (\varphi \leftrightarrow \wedge \nabla(\varphi))$. Since T is universal and has the amalgamation property, for each ψ in $\nabla(\varphi)$ there is a quantifier-free formula $\rho(v_1, \cdots, v_n)$ such that $T \vdash \forall v_1 \cdots \forall v_n\ (\varphi \to \rho)$ and $T \vdash \forall v_1 \cdots \forall v_n\ (\rho \to \psi)$ (see reference [1]). Let $\bar{\nabla}(\varphi) = \{\rho(v_1, \cdots, v_n) : \rho$ is a quantifier-free formula and $T \vdash \varphi \to \rho\}$.

Each existentially complete structure for T is a model of T (since T is universal), so it follows that each existentially complete structure for T satisfies the formula $\forall v_1 \cdots \forall v_n \ (\varphi \leftrightarrow \wedge \bar{\nabla}(\varphi))$. In other words, $\bar{\nabla}(\varphi)$ is existentially principal for $\mathscr{E}_T$. By hypothesis, $\bar{\nabla}(\varphi)$ is self-generated for T. Let ρ_0 be a conjunction of formulas from $\bar{\nabla}(\varphi)$ such that $T \vdash \rho_0 \rightarrow \rho$ for each ρ in $\bar{\nabla}(\varphi)$. Again, since each existentially complete structure for T is a model of T, each existentially complete structure for T must satisfy the formula $\forall v_1 \cdots \forall v_n$ $(\varphi \leftrightarrow \rho_0)$. Hence, the theory $\mathrm{Th}(\mathscr{E}_T)$ is model-complete and is the model-completion of T.

COROLLARY 1. *Under the assumptions of Theorem* 2, *the theory of* $\mathscr{E}_T$ *permits elimination of quantifiers.*

PROOF. This is obvious from the proof of Theorem 2. Also, it follows from a theorem of A. Robinson [49, p. 236, theor. 9.2.19].

There are comparable results concerning universal types. However, these results must use the inductive-companion of the theory rather than the theory itself.

DEFINITION 4. For any theory T, the *inductive-companion* T^2 of T is defined by $T^2 = \{\varphi : \varphi$ is an $\forall\exists$ sentence and $T_\forall \cup \{\varphi\}$ is mutually model-consistent with $T_\forall$ (or equivalently, with T)$\}$, where $T_\forall$ denotes the set of universal consequences of T [33; 35, p. 105].

The set T^2 is a consistent set of sentences. If T has a model-companion T^*, then T^* and T^2 are logically equivalent. More generally, T^2 is the $\forall\exists$ theory of the class $\mathscr{E}_T$.

THEOREM 3. *Let T be an arbitrary (first-order) theory. If each universal type which is existentially principal for the class of existentially complete structures for T is self-generated for T^2, then T^2 is a model-companion for T.*

Conversely, if T^2 is a model-companion for T, then each universal type which is principal for the class of existentially complete structures for T is self-generated for T^2.

PROOF. First, assume that each universal type which is existentially principal for the class $\mathscr{E}_T$ is self-generated for T^2. Let $\varphi(v_1, \cdots, v_n)$ be an existential formula, and let $\nabla(\varphi) = \{\psi(v_1, \cdots, v_n) : \psi$ is a universal formula and $T \vdash \varphi \rightarrow \psi\}$. The set $\nabla(\varphi)$ is an existentially principal type for $\mathscr{E}_T$. Consequently, $\nabla(\varphi)$ is self-generated for T^2. Let ψ_0 be a finite conjunction of formulas from $\nabla(\varphi)$ such that $T^2 \vdash \forall v_1 \cdots \forall v_n \ (\psi_0 \rightarrow \psi)$ for each ψ in $\nabla(\varphi)$. Since $\mathscr{E}_T$ is included in the class of models of T^2, each existentially complete structure satisfies the sentence $\forall v_1 \cdots \forall v_n\ (\psi_0 \rightarrow \wedge \nabla(\varphi))$. Also, each existentially complete structure satisfies the sentence $\forall v_1 \cdots \forall v_n\ (\varphi \leftrightarrow \wedge \nabla(\varphi))$. Hence, each existentially complete structure satisfies the sentence $\forall v_1 \cdots \forall v_n\ (\varphi \leftrightarrow \psi_0)$. Since this sentence is $\forall\exists$, it is in T^2. Thus, each existential formula is equivalent in T^2 to a universal formula, so T^2 is model-complete. Since T and T^2 are mutually model-consistent, T^2 is a model-companion for T.

The converse is immediate.

COROLLARY 1. *If there is a universal type which is principal for the class $\mathscr{E}_T$ but is not self-generated for $\mathscr{E}_T$, then T does not have a model-companion.*

COROLLARY 2. (D. Saracino). *The theory of metabelian groups does not have a model-companion.*

PROOF. D. Saracino [63] proved that the type $\{\forall y\, \forall z\, (x^n \neq [y, z]) : n > 0\}$ is principal for the class $\mathscr{E}_T$ with generator $\forall y\, \forall z\, \exists u\, \exists v\, ([y, z] = [x, [u, v]])$ and that this type is not self-generated. Here $[y, z]$ denotes the commutator $y^{-1}z^{-1}yz$ of y and z.

3. Model-companions for universal theories with finite presentations

The class of universal theories with finite presentations and the amalgamation property includes many of the common theories of algebra, for example, groups, abelian groups, torsion-free abelian groups, R-modules, and Boolean algebras. The theories in this class which have model-companions are characterized in Theorem 5 in this section.

Throughout this section T will denote a universal theory.

A model $\mathfrak{M}$ of T generated by elements $a_1, \cdots, a_n$ is said to be *finitely presented* if there is a finite set $P = \{\rho_1(v_1, \cdots, v_n), \cdots, \rho_k(v_1, \cdots, v_n)\}$ of atomic formulas such that $\mathfrak{M}$ satisfies $\rho_i(a_1, \cdots, a_n)$ for $i = 1, \cdots, k$, and, for each atomic formula $\rho(v_1, \cdots, v_n)$, $\mathfrak{M}$ satisfies $\rho(a_1, \cdots, a_n)$ if and only if $T \vdash \forall v_1 \cdots \forall v_n [(\rho_1 \wedge \cdots \wedge \rho_k) \rightarrow \rho]$. The set P is called a *presentation* for $\mathfrak{M}$.

A model $\mathfrak{M}$ of T generated by elements $a_1, \cdots, a_n$ has a finite presentation $P = \{\rho_1, \cdots, \rho_k\}$ if and only if, whenever a model $\mathfrak{M}'$ of T generated by elements $a'_1, \cdots, a'_n$ satisfies $\rho_i(a'_1, \cdots, a'_n)$ for $i = 1, \cdots, k$, then there is a homomorphism from $\mathfrak{M}$ onto $\mathfrak{M}'$ determined by sending a_i to a'_i for $i = 1, \cdots, n$.

The theory T will be said to have *finite presentations* if, for each finite set $P = \{\rho_1(v_1, \cdots, v_n), \cdots, \rho_k(v_1, \cdots, v_n)\}$ (k and n arbitrary) of atomic formulas for which the sentence $\exists v_1 \cdots \exists v_n\ (\rho_1 \wedge \cdots \wedge \rho_k)$ is consistent with T, there is a finitely presented model of T with presentation P.

In the remainder of this section, the sequences $v_1, \cdots, v_n$, $a_1, \cdots, a_n$, etc., may be denoted by $\bar{v}$, $\bar{a}$, etc., respectively. The sequences $v_1, \cdots, v_n$, $v_{n+1}, \cdots, v_m$, $a_1, \cdots, a_n$, $a_{n+1}, \cdots, a_m$, etc., may be denoted by $\tilde{v}$, $\tilde{a}$, etc., respectively. The positive diagram of a structure $\mathfrak{M}$, denoted by $\mathrm{Diag}^+(\mathfrak{M})$, is the set of atomic sentences which are defined and true in $\mathfrak{M}$.

THEOREM 4. *Assume that $P = \{\rho_1(v_1, \cdots, v_n), \cdots, \rho_k(v_1, \cdots, v_n)\}$ is a set of atomic formulas for which the sentence $\exists v_1 \cdots \exists v_n\ (\rho_1 \wedge \cdots \wedge \rho_k)$ is consistent with T. There is a finitely presented model of T with presentation P if and only if, whenever $\sigma_1(v_1, \cdots, v_n), \cdots, \sigma_p(v_1, \cdots, v_n)$ are atomic formulas and*

$$T \vdash \forall v_1 \cdots \forall v_n\ [(\rho_1 \wedge \cdots \wedge \rho_k) \rightarrow (\sigma_1 \vee \cdots \vee \sigma_p)],$$

then $T \vdash \forall v_1 \cdots \forall v_n\ [(\rho_1 \wedge \cdots \wedge \rho_k) \rightarrow \sigma_i)$ for some i between 1 and p inclusive.

PROOF. Let $\rho(v_1, \cdots, v_n)$ be the conjunction of the formulas in P.

First, assume that T has a model $\mathfrak{M}$ generated by elements $a_1, \cdots, a_n$ with presentation P. Suppose that $\sigma_1(\bar{v}), \cdots, \sigma_p(\bar{v})$ were atomic formulas for which

$$(*) \qquad T \vdash \forall \bar{v}\ [\rho \rightarrow (\sigma_1 \vee \cdots \vee \sigma_p)]$$

but $T \nvdash \forall \bar{v}\ [\rho \rightarrow \sigma_i]$ for $i = 1, \cdots, p$. Then there would be for each $i = 1, \cdots, p$ a model $\mathfrak{M}_i$ of T generated by elements $a_{i,1}, \cdots, a_{i,n}$ such that $\mathfrak{M}_i$ satisfied $\rho(a_{i,1}, \cdots, a_{i,n})$ and $\neg \sigma_i(a_{i,1}, \cdots, a_{i,n})$. Since $\mathfrak{M}$ has presentation P, there would be a homomorphism of $\mathfrak{M}$ onto $\mathfrak{M}_i$ sending a_j to $a_{i,j}$ for $j = 1, \cdots, n$, $i = 1, \cdots, p$. Then $\mathfrak{M}$ would have to satisfy $\neg \sigma_i(a_1, \cdots, a_n)$ for $i = 1, \cdots, p$, since positive formulas are preserved under homomorphism. But then $\mathfrak{M}$ would satisfy $\neg \sigma_1(a_1, \cdots, a_n) \wedge \cdots \wedge \neg \sigma_p(a_1, \cdots, a_n)$ contradicting (*). Hence, there is no such collection of atomic formulas $\sigma_1, \cdots, \sigma_p$.

Conversely, assume that whenever $T \vdash \forall \bar{v}\ [\rho \rightarrow (\sigma_1 \vee \cdots \vee \sigma_p)]$ for atomic formulas $\sigma_1(v_1, \cdots, v_n), \cdots, \sigma_p(v_1, \cdots, v_n)$, then $T \vdash \forall \bar{v}\ [\rho \rightarrow \sigma_i]$ for some i between 1 and p inclusive. Augment the language of T by adjoining new constants $a_1, \cdots, a_n$. Let A be the set of atomic sentences in the augmented language. Let $D^+ = \{\theta : \theta$ is in A and $T \vdash \rho(\bar{a}) \rightarrow \theta\}$, and let $D^- = \{\neg \theta : \theta$ is in A and $T \nvdash \rho(\bar{a}) \rightarrow \theta\}$.

The set $T \cup D^+ \cup D^-$ is consistent. To verify this, suppose that it is not consistent. Then there are sentences $\theta_1, \cdots, \theta_r$ in D^+ and sentences $\neg \theta_{r+1}, \cdots, \neg \theta_s$ in D^- such that $T \vdash \neg(\theta_1 \wedge \cdots \wedge \theta_r \wedge \neg \theta_{r+1} \wedge \cdots \wedge \neg \theta_s)$. Then $T \vdash (\theta_1 \wedge \cdots \wedge \theta_r) \rightarrow (\theta_{r+1} \vee \cdots \vee \theta_s)$. Since $\theta_1, \cdots, \theta_r$ are in D^+,

$T \vdash \rho(\bar{a}) \rightarrow (\theta_{r+1} \vee \cdots \vee \theta_s)$. By assumption, $T \vdash \rho(\bar{a}) \rightarrow \theta_i$ for some i between $r+1$ and s inclusive. But this contradicts that $\neg \theta_i$ is in D^-. Hence, $T \cup D^+ \cup D^-$ is consistent. Let $\mathfrak{N}$ be a model of $T \cup D^+ \cup D^-$. Let $\mathfrak{M}$ be the substructure of $\mathfrak{N}$ generated by $a_1, \cdots, a_n$. Since T is universal, $\mathfrak{M}$ is a model of T. Moreover, $\mathrm{Diag}(\mathfrak{M}) = D^+ \cup D^-$. Hence, $\mathfrak{M}$ has presentation P.

COROLLARY 1. *Every universal-Horn theory has finite presentations.*

PROOF. Suppose that $P = \{\rho_1(v_1, \cdots, v_n), \cdots, \rho_k(v_1, \cdots, v_n)\}$ is a set of atomic formulas for which $\exists v_1 \cdots \exists v_n \ (\rho_1 \wedge \cdots \wedge \rho_k)$ is consistent with a universal-Horn theory T. Suppose that $\sigma_1(v_1, \cdots, v_n), \cdots, \sigma_p(v_1, \cdots, v_n)$ are atomic formulas for which

$$T \vdash \forall \bar{v} \, [(\rho_1 \wedge \cdots \wedge \rho_k) \rightarrow (\sigma_1 \vee \cdots \vee \sigma_p)]$$

but

$$T \nvdash \forall \bar{v} \, [(\rho_1 \wedge \cdots \wedge \rho_k) \rightarrow \sigma_i] \quad \text{for} \quad i = 1, \cdots, p.$$

Let $\mathfrak{M}_i$ for $i = 1, \cdots, p$ be the models described in the preceding proof for this situation. Let $\bar{\mathfrak{M}}$ be the submodel of $\Pi_{i=1}^{p} \mathfrak{M}_i$ generated by the elements $\boldsymbol{a}_1 = (a_{1,1}, a_{2,1}, \cdots, a_{p,1}), \cdots \boldsymbol{a}_n = (a_{1,n}, a_{2,n}, \cdots, a_{p,n})$. Then $\bar{\mathfrak{M}}$ is a model of T and $\bar{\mathfrak{M}}$ satisfies

$$\rho_1(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n) \wedge \cdots \wedge \rho_k(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n) \wedge \neg \sigma_1(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n) \wedge \cdots \wedge \neg \sigma_p(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$$

contradicting the choice of $\sigma_1, \cdots, \sigma_p$.

DEFINITION 5. A theory with finite presentations will be said to be *coherent* if each finitely generated submodel of a finitely presented model is finitely presented itself.

A theory is said to have the *congruence extension property* if whenever $\mathfrak{M}$, $\mathfrak{N}$, and $\mathfrak{N}'$ are models of the theory and $g : \mathfrak{N} \rightarrow \mathfrak{M}$ is an injective homomorphism and $h : \mathfrak{N} \rightarrow \mathfrak{N}'$ is a surjective homomorphism, then there is a model $\mathfrak{M}'$ of the theory, an injective homomorphism $\bar{g} : \mathfrak{N}' \rightarrow \mathfrak{M}'$, and a surjective homomorphism $\bar{h} : \mathfrak{M} \rightarrow \mathfrak{M}'$ such that $\bar{g} \circ h = \bar{h} \circ g$.

A theory is said to have the *homomorphism lifting property* (also known in the literature as the *injections transferable property*) if whenever $\mathfrak{M}$, $\mathfrak{N}$, and $\mathfrak{N}'$ are models of the theory, $g : \mathfrak{N} \rightarrow \mathfrak{M}$ is an injective homomorphism, and $h : \mathfrak{N} \rightarrow \mathfrak{N}'$ is a homomorphism, then there is a model $\mathfrak{M}'$ of the theory, an injective homomorphism $\bar{g} : \mathfrak{N}' \rightarrow \mathfrak{M}'$, and a homomorphism $\bar{h} : \mathfrak{M} \rightarrow \mathfrak{M}'$ such that $\bar{g} \circ h = \bar{h} \circ g$.

If a universal theory has the amalgamation property and the congruence extension property, then it has the homomorphism lifting property. If a universal theory has the homomorphism lifting property, then it has the congruence extension property.

PROPOSITION 1. *If a universal theory with finite presentations has the homomorphism lifting property, then it has the amalgamation property.*

PROOF. Assume that T is a universal theory with finite presentations and the homomorphism lifting property. Suppose that $\mathfrak{M}$, $\mathfrak{M}'$, and $\mathfrak{M}''$ are models of T and that $\mathfrak{M}'$ and $\mathfrak{M}''$ extend $\mathfrak{M}$. It suffices to show that $T \cup \mathrm{Diag}(\mathfrak{M}') \cup \mathrm{Diag}(\mathfrak{M}'')$ is consistent, where the same constant symbols are used in both $\mathrm{Diag}(\mathfrak{M}')$ and $\mathrm{Diag}(\mathfrak{M}'')$ to denote elements of $\mathfrak{M}$ and $\mathrm{Diag}(\mathfrak{M}')$ and $\mathrm{Diag}(\mathfrak{M}'')$ have no other constant symbols in common. Since T has the homomorphism lifting property, $T \cup \mathrm{Diag}(\mathfrak{M}') \cup \mathrm{Diag}^+(\mathfrak{M}'')$ and $T \cup \mathrm{Diag}^+(\mathfrak{M}') \cup \mathrm{Diag}(\mathfrak{M}'')$ are consistent. Suppose that $T \cup \mathrm{Diag}(\mathfrak{M}') \cup \mathrm{Diag}(\mathfrak{M}'')$ is inconsistent. Then there are sentences $\rho_1, \cdots, \rho_q, \neg \rho_{q+1}, \cdots, \neg \rho_r$ in $\mathrm{Diag}(\mathfrak{M}')$ and sentences $\sigma_1, \cdots, \sigma_s, \neg \sigma_{s+1}, \cdots, \neg \sigma_t$ in $\mathrm{Diag}(\mathfrak{M}'')$, where the ρ_i and σ_j are atomic, such that

$$T \vdash \neg [\rho_1 \wedge \cdots \wedge \rho_q \wedge \neg \rho_{q+1} \wedge \cdots \wedge \neg \rho_r \wedge \sigma_1 \wedge \cdots \wedge \sigma_s \wedge \neg \sigma_{s+1} \wedge \cdots \wedge \neg \sigma_t].$$

Then

$$T \vdash [(\rho_1 \wedge \cdots \wedge \rho_q \wedge \sigma_1 \wedge \cdots \wedge \sigma_s) \rightarrow (\rho_{q+1} \vee \cdots \vee \rho_r \vee \sigma_{s+1} \vee \cdots \vee \sigma_t)].$$

Since T has finite presentations, $T \cup \{\rho_1, \cdots, \rho_q, \sigma_1, \cdots, \sigma_s\}$ must imply one of $\rho_{q+1}, \cdots, \rho_r, \sigma_{s+1}, \cdots, \sigma_t$. But none of $\rho_{q+1}, \cdots, \rho_r$ can be implied thusly, because $T \cup \mathrm{Diag}(\mathfrak{M}') \cup \mathrm{Diag}^+(\mathfrak{M}'')$ is consistent. Likewise, none of $\sigma_{s+1}, \cdots, \sigma_t$ can be implied thusly, because $T \cup \mathrm{Diag}^+(\mathfrak{M}') \cup \mathrm{Diag}(\mathfrak{M}'')$ is consistent. Hence, $T \cup \mathrm{Diag}(\mathfrak{M}') \cup \mathrm{Diag}(\mathfrak{M}'')$ is consistent.

DEFINITION 6. A theory T will be said to have the *conservative homomorphism lifting property* (*conservative congruence extension property*) if whenever $\mathfrak{N}$ and $\mathfrak{M}$ are models of T, $\mathfrak{M}$ extends $\mathfrak{N}$, $\psi(v_1, \cdots, v_n, v_{n+1}, \cdots, v_m)$ is a conjunction of negated atomic formulas, and $\mathfrak{M}$ satisfies $\psi(a_1, \cdots, a_n, a_{n+1}, \cdots, a_m)$ for elements $a_1, \cdots, a_n$ of $\mathfrak{N}$ and elements $a_{n+1}, \cdots, a_m$ in $\mathfrak{M}$ but not $\mathfrak{N}$, then there is a quantifier-free formula $\chi(v_1, \cdots, v_n)$ such that $T \cup \mathrm{Diag}^+(\mathfrak{M}) \vdash \psi(a_1, \cdots, a_n, a_{n+1}, \cdots, a_m) \rightarrow \chi(a_1, \cdots, a_n)$ and if $h : \mathfrak{N} \rightarrow \mathfrak{N}'$ is a homomorphism (surjective homomorphism, resp.) and $\mathfrak{N}'$ satisfies $\chi(h(a_1), \cdots, h(a_n))$, then there is a model $\mathfrak{M}'$ of T which extends $\mathfrak{N}'$ and a homomorphism (surjective homomorphism, resp.) $\bar{h} : \mathfrak{M} \rightarrow \mathfrak{M}'$ such that the restriction of $\bar{h}$ to $\mathfrak{N}$ coincides with h and $\mathfrak{M}'$ satisfies $\psi(h(a_1),\cdots,h(a_n),\bar{h}(a_{n+1}),\cdots,\bar{h}(a_m))$.

DEFINITION 7. A theory T will be said to have the *conservative homomorphism lifting property for finite presentations* (*conservative congruence extension property for finite presentations*, respectively) if the preceding definition holds with the additional restrictions that $\mathfrak{M}$ is generated by $a_1,\cdots,a_n,a_{n+1},\cdots,a_m$ with some finite presentation P and $\mathfrak{N}$ is generated by $a_1,\cdots,a_n$.

A theory with the amalgamation property or the homomorphism lifting property has the conservative homomorphism lifting property (for finite presentations) if and only if it has the conservative congruence extension property (for finite presentations, respectively).

A theory with finite presentations has the conservative homomorphism lifting property (for finite presentations) if and only if it has the amalgamation property and the conservative congruence extension property (for finite presentations, respectively).

The main theorem of this paper is the following.

THEOREM 5. *Assume that T is a universal theory with finite presentations, at least one constant symbol, and the amalgamation property. The theory T has a model-completion, or equivalently a model-companion, if and only if T is coherent and has the conservative congruence extension property for finite presentations.*

PROOF. (i) Assume that T has a model-completion T^*. Let $\mathfrak{N}$ be a finitely generated submodel of a finitely presented model $\mathfrak{M}$ of T. Assume that $\mathfrak{N}$ is generated by elements $a_1,\cdots,a_n$ and that $\mathfrak{M}$ is generated by the elements $a_1,\cdots,a_n$ together with elements $a_{n+1},\cdots,a_m$ (not in $\mathfrak{N}$) with presentation $P=\{\rho_1(v_1,\cdots,v_n,v_{n+1},\cdots,v_m),\cdots,\rho_k(v_1,\cdots,v_n,v_{n+1},\cdots,v_m)\}$. Let ρ be the conjunction $\rho_1\wedge\cdots\wedge\rho_k$. Let $\varphi(v_1,\cdots,v_n)$ be the formula $\exists v_{n+1}\cdots\exists v_m\,\rho$.

Since T^* is the model-completion of T, T^* permits elimination of quantifiers (theor. 1 and corol. 1 to theor. 2, or theor. 9.2.19 of reference 49, p. 236). Let $\theta(v_1,\cdots,v_n)$ be a quantifier-free formula such that $T^*\vdash\forall v_1\cdots\forall v_n\,(\varphi\leftrightarrow\theta)$. One may assume without loss of generality that θ is a disjunction $\theta_1\vee\cdots\vee\theta_p$ where each θ_i is a conjunction of atomic formulas and negated atomic formulas. Since $\mathfrak{M}$ is included in a model of T^*, $\mathfrak{N}$ must satisfy $\theta_i(\bar{a})$ for some i. Let $\sigma_1,\cdots,\sigma_r$ be the positive conjuncts of θ_i for this i. Let $\mathfrak{N}'$ be the model of T generated by elements $a_1',\cdots,a_n'$ with presentation $\{\sigma_1,\cdots,\sigma_r\}$. There is a homomorphism of $\mathfrak{N}'$ onto $\mathfrak{N}$ sending a_j' to a_j. Therefore, $\mathfrak{N}'$ satisfies $\theta_i(a_1',\cdots,a_n')$ for the i specified above. Since $\mathfrak{N}'$ is included in a model of T^*, there is a model $\mathfrak{M}'$ of T which extends $\mathfrak{N}'$, is generated by the elements $a'_1, \cdots, a'_n$ together with elements $a_{n+1}, \cdots, a'_m$, and satisfies $\rho(\bar{a}')$. There is a homomorphism of $\mathfrak{M}$ onto $\mathfrak{M}'$ sending a_j to a'_j, since P is a presentation of $\mathfrak{M}$. Thus, $\mathfrak{N}$ and $\mathfrak{N}'$ are isomorphic. Hence, $\{\sigma_1, \cdots, \sigma_r\}$ is a finite presentation for $\mathfrak{N}$.

Now suppose that $\psi(v_1, \cdots, v_n, v_{n+1}, \cdots, v_m)$ is a conjunction of negated atomic formulas and that $\mathfrak{M}$ satisfies $\psi(\bar{a})$. Let $\chi(v_1, \cdots, v_n)$ be a quantifier-free formula for which

$$T^* \vdash \forall v_1 \cdots \forall v_n [(\exists v_{n+1} \cdots \exists v_m (\rho(\bar{v}) \wedge \psi(\bar{v}))) \leftrightarrow \chi(\bar{v})].$$

Since T and T^* have the same universal consequences, $T \cup \mathrm{Diag}^+(\mathfrak{M}) \vdash \psi(\bar{a}) \rightarrow \chi(\bar{a})$. Suppose that $\mathfrak{N}''$ is a model of T generated by elements $a''_1, \cdots, a''_n$ and that there is a homomorphism of $\mathfrak{N}$ onto $\mathfrak{N}''$ sending a_j to a''_j. Suppose further that $\mathfrak{N}''$ satisfies $\chi(\bar{a}'')$. The structure $\mathfrak{N}''$ has an extension $\mathfrak{M}'''$ which is a model of T^*. There are elements $a''_{n+1}, \cdots, a''_m$ of $\mathfrak{M}'''$ for which $\mathfrak{M}'''$ satisfies $\rho(\bar{a}'') \wedge \psi(\bar{a}'')$. Let $\mathfrak{M}''$ be the submodel of T generated by $a''_1, \cdots, a''_n, a''_{n+1}, \cdots, a''_m$. There is a homomorphism of $\mathfrak{M}$ onto $\mathfrak{M}''$ sending a_j to a''_j, because P is a presentation of $\mathfrak{M}$.

Thus, T is coherent and has the conservative congruence extension property for finite presentations.

(ii) Assume that T is coherent and has the conservative congruence extension property for finite presentations. A model-completion for T will be constructed.

Suppose that $\varphi(v_1, \cdots, v_n)$ is a primitive existential formula such that $\exists v_1 \cdots \exists v_n\ \varphi$ is consistent with T. Then φ has the form

$$\begin{aligned}(1)\qquad & \exists v_{n+1} \cdots \exists v_m [\rho_1(v_1, \cdots, v_n, v_{n+1}, \cdots, v_m) \wedge \cdots \wedge \\ & \rho_k(v_1, \cdots, v_n, v_{n+1}, \cdots, v_m) \wedge \neg\, \sigma_1(v_1, \cdots, v_n, v_{n+1}, \cdots, v_m) \wedge \cdots \wedge \\ & \neg\, \sigma_p(v_1, \cdots, v_n, v_{n+1}, \cdots, v_m)]\end{aligned}$$

where the ρ_i and σ_j are atomic. The theory T has a model $\mathfrak{M}$ generated by elements $a_1, \cdots, a_n, a_{n+1}, \cdots, a_m$ with presentation $\{\rho_1, \cdots, \rho_k\}$. The model $\mathfrak{M}$ must satisfy $\neg\, \sigma_1(\bar{a}) \wedge \cdots \wedge \neg\, \sigma_p(\bar{a})$. Let $\mathfrak{N}$ be the submodel generated by $a_1, \cdots, a_n$. Since T is coherent, $\mathfrak{N}$ is finitely presented. Let $\theta(v_1, \cdots, v_n)$ be a conjunction of a finite presentation for $\mathfrak{N}$. Let $\psi(v_1, \cdots, v_n, v_{n+1}, \cdots, v_m)$ be the formula $\neg\, \sigma_1(\bar{v}) \wedge \cdots \wedge \neg\, \sigma_p(\bar{v})$, and let $\chi(v_1, \cdots, v_n)$ be the formula described in the definition of the conservative congruence extension property for this choice of $\mathfrak{M}$, $\mathfrak{N}$, and ψ. Let $A(\varphi)$ be the formula

$$\forall v_1 \cdots \forall v_n [\varphi(v_1, \cdots, v_n) \leftrightarrow (\theta(v_1, \cdots, v_n) \wedge \chi(v_1, \cdots, v_n))].$$

Let $T^* = T \cup \{A(\varphi) : \varphi(v_1, \cdots, v_n)$ is a primitive existential formula for which

$\exists v_1 \cdots \exists v_n \varphi$ is consistent with $T\}$. In order to verify that T^* is the model-completion of T, it will suffice according to Theorem A to show that each model of T is included in a model of T^* and that T^* is model-complete.

Let $\mathfrak{M}_1$ be an existentially complete structure for T. Let $\varphi(v_1, \cdots, v_n)$ be a primitive existential formula for which $\exists v_1 \cdots \exists v_n \varphi$ is consistent with T. Assume that φ has the form in (1). Suppose that $\mathfrak{M}_1$ satisfies $\varphi(a'_1, \cdots, a'_n)$ for some elements $a'_1, \cdots, a'_n$. Let $a'_{n+1}, \cdots, a'_m$ be elements of $\mathfrak{M}_1$ for which $\mathfrak{M}_1$ satisfies $\rho_1(\tilde{a}') \wedge \cdots \wedge \rho_k(\tilde{a}') \wedge \neg \sigma_1(\tilde{a}') \wedge \cdots \wedge \neg \sigma_p(\tilde{a}')$. Let $\mathfrak{M}'$ be the submodel generated by $a'_1, \cdots, a'_n, a'_{n+1}, \cdots, a'_m$, and let $\mathfrak{N}'$ be the submodel generated by $a'_1, \cdots, a'_n$. The model $\mathfrak{M}'$ is a homomorphic image of $\mathfrak{M}$. The same homomorphism maps $\mathfrak{N}$ onto $\mathfrak{N}'$. Since θ is the conjunction of a presentation of $\mathfrak{N}$, the model $\mathfrak{N}'$ satisfies $\theta(a'_1, \cdots, a'_n)$. The formula χ was chosen so that $T \cup \mathrm{Diag}^+(\mathfrak{M}) \vdash \psi(a_1, \cdots, a_n, a_{n+1}, \cdots, a_m) \rightarrow \chi(a_1, \cdots, a_n)$. Since $\mathfrak{M}'$ is a model of $\mathrm{Diag}^+(\mathfrak{M})$, $\mathfrak{M}'$ satisfies $\chi(a'_1, \cdots, a'_n)$. Thus, $\mathfrak{M}'$ satisfies $\theta(a'_1, \cdots, a'_n) \wedge \chi(a'_1, \cdots, a'_n)$, so $\mathfrak{M}_1$ satisfies this (quantifier-free) sentence also.

Suppose now that $\mathfrak{M}_1$ satisfies $\theta(a''_1, \cdots, a''_n) \wedge \chi(a''_1, \cdots, a''_n)$ for some elements $a''_1, \cdots, a''_n$. Let $\mathfrak{N}''$ be the submodel generated by $a''_1, \cdots, a''_n$. Since $\mathfrak{N}''$ satisfies $\theta(a''_1, \cdots, a''_n)$, there is a homomorphism of $\mathfrak{N}$ onto $\mathfrak{N}''$ sending a_i to a''_i. Since $\mathfrak{N}''$ satisfies $\chi(a''_1, \cdots, a''_n)$, there is a model $\mathfrak{M}''$ of T which extends $\mathfrak{N}''$, is generated by the elements $a''_1, \cdots, a''_n$ together with elements $a''_{n+1}, \cdots, a''_m$, and satisfies $\psi(\tilde{a}'')$, and there is a homomorphism from $\mathfrak{M}$ onto $\mathfrak{M}''$ sending a_i to a''_i. Therefore, $\mathfrak{M}''$ satisfies $\rho_1(\tilde{a}'') \wedge \cdots \wedge \rho_k(\tilde{a}'') \wedge \neg \sigma_1(\tilde{a}'') \wedge \cdots \wedge \neg \sigma_p(\tilde{a}'')$. There is a model $\mathfrak{M}_2$ of T which extends both $\mathfrak{M}_1$ and $\mathfrak{M}''$, since T has the amalgamation property. The model $\mathfrak{M}_2$ satisfies $\varphi(a''_1, \cdots, a''_n)$. Since $\mathfrak{M}_1$ is existentially complete, it satisfies $\varphi(a''_1, \cdots, a''_n)$ also. Thus, $\mathfrak{M}_1$ is a model of $A(\varphi)$.

Thus, each existentially complete structure for T is a model of T^*. Since each model of T is included in an existentially complete structure, T^* is mutually model-consistent with T.

If $\varphi(v_1, \cdots, v_n)$ is an existential sentence such that $\exists v_1 \cdots \exists v_n \varphi$ is inconsistent with T, then T implies $\forall v_1 \cdots \forall v_n [\varphi \leftrightarrow \neg(c = c)]$, where c is a constant symbol occurring in T. Thus, each existential formula is equivalent in T^* to a quantifier-free formula. Hence, T^* is model-complete.

Corollary 1. *Assume that T is a universal theory with finite presentations, at least one constant symbol, and the amalgamation property. If the language of T contains at most finitely many predicate symbols (but may have arbitrarily many constant symbols and function symbols) and every finitely generated model of T is finite, then T has a model-completion.*

PROOF. Since every finitely generated model is finite, T is coherent. Moreover, T has the conservative congruence extension property for finite presentations, because each finitely generated structure has only a finite number of homomorphic images.

COROLLARY 2. *The theory of Boolean algebras has a model-completion.*

PROOF. The theory of Boolean algebras satisfies the hypotheses of Corollary 1.

COROLLARY 3 (G. Sabbagh [61], R. Cusin and J. R. Pabion [22], and D. Saracino). *The theory of p-rings for each prime p has a model-completion.*

PROOF. G. Sabbagh [61] proved that the theory of p-rings for a prime p has the amalgamation property. Consequently, this theory satisfies the hypotheses of Corollary 1.

COROLLARY 4 (P. Eklof and G. Sabbagh [25]). *The theory of groups does not have a model-companion.*

PROOF. The theory of groups satisfies the hypotheses of Theorem 5. It is well known from Higman's Theorem that there is a finitely presented group with a finitely generated subgroup which is recursively presented but not finitely presented. Consequently, this theory is not coherent and so does not have a model-companion.

The proof of the preceding corollary is interesting, because all previous proofs of the corollary showed essentially that the theory of groups did not have the conservative congruence extension property for finite presentations.

The verification of the conservative congruence extension property for finite presentations is tedious for most interesting theories with this property. Examples are the theory of abelian groups and the theory of R-modules. This has led to the introduction of the model-cancellation property defined below. The verification of this property is usually straightforward.

DEFINITION 8. A theory T will be said to have the *model-cancellation property* if whenever $\mathfrak{N}$ and $\mathfrak{M}$ are models of T, $\mathfrak{M}$ extends $\mathfrak{N}$, ρ is a conjunction of atomic sentences defined in $\mathfrak{N}$ such that $T \cup \mathrm{Diag}^+(\mathfrak{N}) \cup \{\rho\}$ is consistent, σ is an atomic sentence defined in $\mathfrak{M}$ but not true in $\mathfrak{M}$, and $T \cup \mathrm{Diag}^+(\mathfrak{M}) \vdash \rho \rightarrow \sigma$, then there is a positive, quantifier-free sentence χ defined in $\mathfrak{N}$ such that $T \cup \mathrm{Diag}^+(\mathfrak{M}) \vdash \chi \leftrightarrow \sigma$.

Of course, a model-cancellation property for finite presentations could be defined in a manner analogous to that for the conservative congruence extension property for finite presentations.

THEOREM 6. *Assume that T is a universal theory with finite presentations, at least one constant symbol, and the congruence extension property. If T has the model-cancellation property, then T has the conservative congruence extension property.*

PROOF. Assume that T has the model-cancellation property. Let $\psi(v_1,\cdots,v_n,v_{n+1},\cdots,v_m)$ be a conjunction

$$\neg\sigma_1(v_1,\cdots,v_n,v_{n+1},\cdots,v_m)\wedge\cdots\wedge\neg\sigma_o(v_1,\cdots,v_n,v_{n+1},\cdots,v_m)$$

of negated atomic formulas $\neg\sigma_1,\cdots,\neg\sigma_p$. Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are models of T, $\mathfrak{M}$ extends $\mathfrak{N}$, $a_1,\cdots,a_n$ are elements of $\mathfrak{N}$, $a_{n+1},\cdots,a_m$ are elements of $\mathfrak{M}$ but not $\mathfrak{N}$, and $\mathfrak{M}$ satisfies $\psi(a_1,\cdots,a_n,a_{n+1},\cdots,a_m)$.

Consider the formula σ_1. Choose a formula χ_1 as follows.

Case 1. There is a conjuction ρ of atomic sentences defined in $\mathfrak{N}$ such that $T\cup\mathrm{Diag}^+(\mathfrak{M})\cup\{\rho\}$ is consistent and implies $\sigma_1(\tilde a)$. Since T has the model-cancellation property, there is a positive, quantifier-free formula $\chi_1(v_1,\cdots,v_n)$ such that

$$T\cup\mathrm{Diag}^+(\mathfrak{M})\vdash\chi_1(a_1,\cdots,a_n)\leftrightarrow\sigma_1(a_1,\cdots,a_n,a_{n+1},\cdots,a_m).$$

Case 2. There is no such formula ρ. Choose χ_1 to be the formula $\neg(c=c)$ where c is a constant symbol occurring in T.

Choose formulas $\chi_2(v_1,\cdots,v_n),\cdots,\chi_p(v_1,\cdots,v_n)$ in a similar manner. Let $\chi(v_1,\cdots,v_n)$ be the formula $\neg\chi_1\wedge\cdots\wedge\neg\chi_p$. Clearly, by the choice of the χ_i, $T\cup\mathrm{Diag}^+(\mathfrak{M})\vdash\psi(a_1,\cdots,a_n,a_{n+1},\cdots,a_m)\rightarrow\chi(a_1,\cdots,a_n)$.

Suppose now that $h:\mathfrak{N}\rightarrow\mathfrak{N}'$ is a homomorphism of $\mathfrak{N}$ onto $\mathfrak{N}'$ and that $\mathfrak{N}'$ satisfies $\chi(h(a_1),\cdots,h(a_n))$. Interpret the constants in the language of $\mathfrak{M}$ which name elements of $\mathfrak{N}$ to name the corresponding images of these elements under h also. Since T has the congruence extension property, $T\cup\mathrm{Diag}^+(\mathfrak{M})\cup\mathrm{Diag}(\mathfrak{N}')$ is consistent. Suppose that this set of sentences implies $\neg\psi(\tilde a)$. Then there are atomic sentences $\theta_1,\cdots,\theta_k$ in $\mathrm{Diag}^+(\mathfrak{M})$, and atomic sentences $\rho_1,\cdots,\rho_r$ and negated atomic sentences $\neg\rho_{r+1},\cdots,\neg\rho_s$ in $\mathrm{Diag}(\mathfrak{N}')$ such that

$$T\vdash[(\theta_1\wedge\cdots\wedge\theta_k\wedge\rho_1\wedge\cdots\wedge\rho_r\wedge\neg\rho_{r+1}\wedge\cdots\wedge\neg\rho_s)\rightarrow(\sigma_1(\tilde a)\vee\cdots\vee\sigma_p(\tilde a))].$$

Then

$$T\vdash[(\theta_1\wedge\cdots\wedge\theta_k\wedge\rho_1\wedge\cdots\wedge\rho_r)\rightarrow(\rho_{r+1}\vee\cdots\vee\rho_s\vee\sigma_1(\tilde a)\vee\cdots\vee\sigma_p(\tilde a))].$$

According to Theorem 4, $T \cup \{\theta_1, \cdots, \theta_k, \rho_1, \cdots, \rho_r\}$ implies one of $\rho_{r+1}, \cdots, \rho_s$, $\sigma_1(\bar{a}), \cdots, \sigma_p(\bar{a})$. But none of $\rho_{r+1}, \cdots, \rho_s$ could be implied, since $T \cup \mathrm{Diag}^+(\mathfrak{M}) \cup \mathrm{Diag}(\mathfrak{N}')$ is consistent. Therefore, $\sigma_i(\bar{a})$ for some i between 1 and p inclusive is implied. Consequently, $T \cup \mathrm{Diag}^+(\mathfrak{M}) \vdash (\rho_1 \wedge \cdots \wedge \rho_r) \rightarrow \sigma_i(\bar{a})$. Therefore, Case 1 above was used to choose the corresponding χ_i, so $T \cup \mathrm{Diag}^+(\mathfrak{M}) \vdash \chi_i(a_1, \cdots, a_n) \leftrightarrow \sigma_i(a_1, \cdots, a_n, a_{n+1}, \cdots, a_m)$. Since $\neg \chi_i(\bar{a})$ is in $\mathrm{Diag}(\mathfrak{N}')$, $T \cup \mathrm{Diag}^+(\mathfrak{M}) \cup \mathrm{Diag}(\mathfrak{N}') \vdash \neg \sigma_i(\bar{a})$. But then $T \cup \mathrm{Diag}^+(\mathfrak{M}) \cup \mathrm{Diag}(\mathfrak{N}')$ implies both $\sigma_i(\bar{a})$ and $\neg \sigma_i(\bar{a})$, which contradicts the consistency of this set of sentences.

Hence, $T \cup \mathrm{Diag}^+(\mathfrak{M}) \cup \mathrm{Diag}(\mathfrak{N}') \cup \{\psi(a_1, \cdots, a_n, a_{n+1}, \cdots, a_m)\}$ is consistent. Therefore, there is a model $\mathfrak{M}'$ of T which extends $\mathfrak{N}'$ and a homomorphism $\bar{h}$ of $\mathfrak{M}$ onto $\mathfrak{M}'$ such that the restriction of $\bar{h}$ to $\mathfrak{N}$ is just h and $\mathfrak{M}'$ satisfies $\psi(h(a_1), \cdots, h(a_n), \bar{h}(a_{n+1}), \cdots, \bar{h}(a_m))$.

Corollary 1 (P. Eklof and G. Sabbagh [25]). *The theory of R-modules has a model-completion if and only if R is coherent.*

Proof. The theory of R-modules has finite presentations, at least one constant symbol, the amalgamation property, and the congruence extension property, and therefore the homomorphism lifting property; all this is well known. Moreover, the theory of R-modules has the model-cancellation property. To verify this, suppose that $\mathfrak{N}$ is a submodule of an R-module $\mathfrak{M}$, $\rho_1, \cdots, \rho_q$ are atomic sentences defined in $\mathfrak{N}$, σ is an atomic sentence defined in $\mathfrak{M}$ but not true in $\mathfrak{M}$, and $T \cup \mathrm{Diag}^+(\mathfrak{M}) \vdash (\rho_1 \wedge \cdots \wedge \rho_q) \rightarrow \sigma$. Each ρ_i has the form $\Sigma r_{ij} a_{ij} = \Sigma s_{ik} b_{ik}$ where the r_{ij} and s_{ik} are elements of R and the a_{ij} and b_{ik} are elements of $\mathfrak{N}$. Let $\mathfrak{R}$ be the submodule generated by the elements $\Sigma r_{ij} a_{ij} - \Sigma s_{ik} b_{ik}$ for $i = 1, \cdots, q$. Since the a_{ij} and b_{ik} are elements of $\mathfrak{N}$, $\mathfrak{R}$ is a submodule of $\mathfrak{N}$. The sentence σ has the form $\Sigma r_j a_j = \Sigma s_k b_k$ where the r_j and s_k are elements of R and the a_j and b_k are elements of $\mathfrak{M}$. Since $T \cup \mathrm{Diag}^+(\mathfrak{M}) \vdash (\rho_1 \wedge \cdots \wedge \rho_q) \rightarrow \sigma$, there is an element c of $\mathfrak{R}$ such that $\mathfrak{M}$ satisfies $\Sigma r_j a_j = \Sigma s_k b_k + c$. Then $T \cup \mathrm{Diag}^+(\mathfrak{M}) \vdash (\Sigma r_j a_j = \Sigma s_k b_k) \leftrightarrow (c = 0)$.

One of the equivalent conditions defining a coherent ring [11, pp. 62–63, problems 11–12] is that each finitely generated submodule of a finitely presented module is finitely presented itself. Hence, the corollary follows from Theorems 5 and 6.

4. Further applications of definable, general elementary properties

When a theory does not have a model-companion, a series of questions arises concerning the class of existentially complete structures and its various subclasses. Definability of general elementary properties is usually relevant to the resolution of these questions. One question is whether every existentially complete structure is both finitely generic and infinitely generic; a weaker question is whether the finite forcing companion coincides with the infinite forcing companion (see [35] or [53] for definitions). The following theorems indicate the relevance of definability for these questions.

THEOREM 7. *Assume that T is a countable, first order theory. If Δ is a universal type which is not existentially generated, then there is an existentially complete structure, in fact, a finitely generic structure which omits this type.*

PROOF. Augment the language of T with an infinite set $A = \{a_m : m < \omega\}$ of new constant symbols. Assume that Δ is an n-type. Let $\{\zeta_m : m < \omega\}$ be an enumeration of all n-tuples of constants in the augmented language. Let $\{\varphi_m : m < \omega\}$ be an enumeration of all the sentences in the augmented language. Construct a complete sequence of forcing conditions as follows.

Step 0. Let $P_0 = \{a_0 = a_0\}$.

Step $2m+1$. If P_{2m} finitely forces φ_m or $\neg\varphi_m$, then let $P_{2m+1} = P_{2m}$. Otherwise, there is a condition Q containing P_{2m} which does finitely force φ_m; let $P_{2m+1} = Q$.

Step $2m+2$. Let $\exists\bar{v}(\wedge P_{2m+1})$ be the formula obtained by replacing distinct constants occurring in P_{2m+1} but not in ζ_m by distinct variables, forming the conjunction of the resulting formulas, and then existentially quantifying over the variables introduced. Since Δ is not existentially principal, there is a formula $\psi(v_1, \cdots, v_n)$ in Δ such that $T \nvdash \exists\bar{v}(\wedge P_{2m+1}) \rightarrow \psi(c_1, \cdots, c_n)$ where $\zeta_m = (c_1, \cdots, c_n)$. We may assume that $\neg\psi(c_1, \cdots, c_n)$ is a disjunction of primitive sentences, namely $\chi_1(c_1, \cdots, c_n) \vee \cdots \vee \chi_p(c_1, \cdots, c_n)$. Then $\chi_i(c_1, \cdots, c_n)$ for some i is consistent with $T \cup P_{2m+1}$. Substitute distinct constants from A which do not occur in P_{2m+1} or $\chi_i(c_1, \cdots, c_n)$ for the variables occurring in the matrix of $\chi_i(c_1, \cdots, c_n)$. Let P_{2m+2} consist of the conjuncts of the resulting matrix together with the formulas in P_{2m+1}. Then P_{2m+2} is consistent with T, so P_{2m+2} is a condition containing P_{2m+1}.

The sequence of conditions $P_0 \subseteq P_1 \subseteq \cdots$ is a complete sequence of forcing conditions. Hence, it determines a unique, finitely generic structure. That this structure omits the type Δ is evident from the even numbered steps.

THEOREM 8. *Assume that T is a countable theory with the joint embedding property. If there is a universal type which is principal but not existentially principal for the class of existentially complete structures for T and is realized in one of these* *structures, then the finite forcing companion T^f and the infinite forcing companion T^F are distinct, complete theories.*

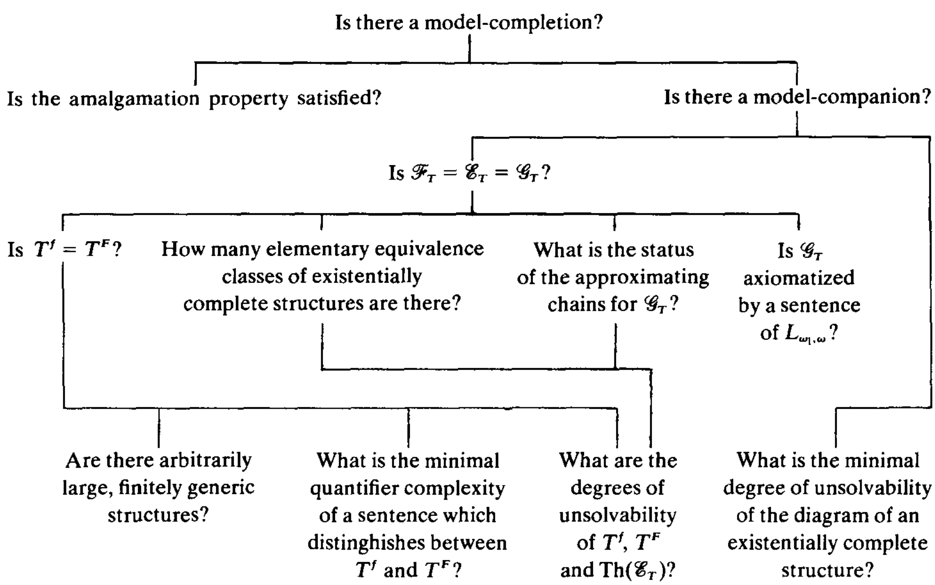
PROOF. Since T has the joint embedding property, both T^f and T^F are complete theories. Let $\varphi(\bar{v})$ be a generator for the type mentioned in the hypotheses. This type must be realized in some infinitely generic structure, so T^F includes the sentence $\exists \bar{v} \varphi(\bar{v})$. On the other hand, this type cannot be existentially generated for T, so there is a finitely generic structure which omits this type. Hence, T^f contains the sentence $\forall \bar{v} \neg \varphi(\bar{v})$.

Types which satisfy the hypotheses of the preceding theorem have been found for the theories of groups [35, 72], metabelian groups [63], nilpotent groups (D. Saracino), commutative rings [17], division rings [35, 72], and arithmetic [34, 35].

Diagram I presents some of the questions which arise for a theory without a model-companion. The diagram also indicates which questions are subordinate to others. Whenever a question has a negative answer, the questions beneath it remain to be answered.

DIAGRAM I

QUESTIONS CONCERNING EXISTENTIALLY COMPLETE STRUCTURES



The definability of general elementary properties and, in some cases, the resulting interpretation of second order arithmetic have led to answers to most of these questions for some theories. Consider, for example, the question of the number of elementary equivalence classes of existentially complete structures. The following theorem [35, p. 130; 72, p. 53] relates this problem to degrees of unsolvability.

The set of Gödel numbers of a countable set S of formulas will be denoted by $\ulcorner S \urcorner$.

THEOREM C. *Assume that* $\ulcorner T \urcorner$ *is an arithmetical set. If the set* $\{\ulcorner T' \urcorner : T' = \mathrm{Th}(\mathfrak{M})$ *for some* $\mathfrak{M}$ *in* $\mathscr{E}_T\}$ *has cardinality less than* $2^{\aleph_0}$, *then the set is countable, each member of the set is hyperarithmetical, and* $\mathrm{Th}(\mathscr{E}_T)$ *is hyperarithmetical.*

The set $\ulcorner T^F \urcorner$ for the theories of arithmetic [34, 35], groups [35, 72], and division rings [35, 72] has been shown to have the same degree of unsolvability as the theory of full second order arithmetic. Hence the set of theories of existentially complete structures for these theories has cardinality $2^{\aleph_0}$. Also, $\mathrm{Th}(\mathscr{E}_T)$ for these theories is a complete Π_1^1 set.

In passing, it should be remarked that the cardinality of the set of theories of existentially complete structures is either countable or the continuum regardless of whether $\ulcorner T \urcorner$ is arithmetical [28]. Examples [28, 65, 68] show that this is the only restriction on the cardinality of this set for countable theories.

The theory of groups is an example for which all questions in the diagram are relevant. As previously mentioned, P. Eklof and G. Sabbagh [25] proved in 1969 that there was no model-companion. A. Macintyre proved in 1970 that $\mathscr{E}_T \neq \mathscr{G}_T$ and $\mathscr{E}_T \neq \mathscr{F}_T$ [37]. Subsequently, he proved that T^f and T^F are distinguished by an $\forall_4$ sentence. In 1972 the author found a principal existential generator for the quantifier-free type of having infinite order [35, 72]. This leads to the following interpretation of second order arithmetic. Let G be an existentially complete group. This group has an element a of infinite order. The set $\{a^n : n \geqq 0\}$ is definable, and there are definable operations $\oplus$ and $\otimes$ such that $a^n \oplus a^m = a^{n+m}$ and $a^n \otimes a^m = a^{nm}$. Moreover, there is a well determined collection $\mathscr{S}$ of subsets of $\{a^n : n \geqq 0\}$. The set $\{a^n : n \geqq 0\}$ with the operations $\oplus$ and $\otimes$ and the collection $\mathscr{S}$ of subsets forms a definable structure $\mathscr{N}_G$ for second order arithmetic. This second order structure is uniquely determined independently of the choice of the element a of infinite order, because all elements of infinite order are conjugate.

THEOREM D [35, 72]. *If* G *is an infinitely generic group, then* $\mathscr{N}_G$ *is a second order elementary substructure of the standard model* $\mathscr{N}$ *for second order arithmetic.*

Hence, $\ulcorner \mathrm{Th}_2(\mathcal{N}) \urcorner \leqq_1 \ulcorner T^F \urcorner$, *where* Th_2 *denotes the second order theory. In fact,* $\ulcorner \mathrm{Th}_2(\mathcal{N}) \urcorner \equiv_1 \ulcorner T^F \urcorner$.

THEOREM E [35, 72]. *There are* $2^{\aleph_0}$ *elementary equivalence classes of existentially complete groups distinghished by* $\forall_4$ *sentences.*

Analogous results have been obtained for arithmetic [34, 35] and division rings [35, 72].

O. V. Belegradek, a Russian mathematician, has communicated the following, more recent results on existentially complete groups to the author: 1) there are $2^{\aleph_0}$ elementary equivalence classes of existentially complete groups distinguished by $\forall_3$ sentences: 2) T^F and T^f are distinguished by an $\forall_3$ sentence; and 3) an existentially complete group is finitely generic if and only if it is a model of T^f.

Appendix

A list of references according to topic appears below.

1. General theory of forcing, model-companions, and existentially complete structures: for a comprehensive discussion, see [35]; other references: [3, 8, 14, 15, 18–22, 27, 28, 36, 38, 41, 51–55, 62, 64, 65, 68, 71, 72, 73].
2. Arithmetic: [3, 34, 35, 45, 56].
3. Modules: [25, 26, 58–61].
4. Abelian groups: [23, 24, 25].
5. Groups: [25, 37, 38, 40, 42, 35, 72].
6. Division rings: [7–9, 35, 39, 40, 42, 72].
7. Commutative rings: [17].
8. Commutative rings without nilpotent elements: [13, 67].
9. Metabelian groups: [63].
10. Nilpotent groups: [66].
11. Lie algebras: [43, 44].

Remarks added in proof

1) It is implicit in Section 3 that a finitely generated model which has a finite presentation relative to one finite set of generators has a finite presentation relative to any finite set of generators. This is easily verified. Assume that $\mathfrak{M}$ is generated by elements $a_1, \cdots, a_n$ with finite presentation $\{\rho_1(v_1, \cdots, v_n), \cdots, \rho_k(v_1, \cdots, v_n)\}$. Suppose that $b_1, \cdots, b_m$ generate $\mathfrak{M}$ also. Then there are terms $t_1(v_1, \cdots, v_n), \cdots, t_m(v_1, \cdots, v_n)$ such that $b_j = t_j(a_1, \cdots, a_n)$

for $j = 1, \cdots, m$, and there are terms $u_1(v_1, \cdots, v_m), \cdots, u_n(v_1, \cdots, v_m)$ such that $a_i = u_i(b_1, \cdots, b_m)$ for $i = 1, \cdots, n$. A presentation for $\mathfrak{M}$ relative to the generators $b_1, \cdots, b_m$ is

$$\{\rho_i(u_1(v_1, \cdots, v_m), \cdots, u_n(v_1, \cdots, v_m)) : i = 1, \cdots, k\} \cup$$

$$\{v_i = t_i(u_1(v_1, \cdots, v_m), \cdots, u_n(v_1, \cdots, v_m)) : i = 1, \cdots, m\}.$$

2) G. Sabbagh has proven that every universal theory with finite presentations is in fact universal Horn. Thus, a theory is universal with finite presentations if and only if it is universal Horn.

3) The assumption in Theorem 5 that T has finite presentations can be weakened so as to include non-Horn theories such as the theory of integral domains. This improvement will appear in a forthcoming paper by the author in which Theorem 5 is generalized to theories which may not have the amalgamation property.

4) The proof of Corollary 4 to Theorem 5 referred to Higman's Theorem for the existence of a finitely presented group with a finitely generated, nonfinitely presentable subgroup. G. Sabbagh has kindly pointed out that such groups were known prior to the appearance of Higman's Theorem (see Section 4 of G. Baumslag, W. W. Boone and B. H. Neumann, *Some unsolvable problems about elements and subgroups of groups*, Math. Scand. **7** (1959), 191–201).

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